

Topological aspects in a two component Bose condensed system in neutron star

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Abstract

By making use of Duan-Ge's decomposition theory of gauge potential and the Duan's topological current theory proposed by Prof. Duan Yi-Shi, we study a two component superfluid Bose condensed system, which is supposed being realized in the interior of neutron stars in the form of a coexistent neutron superfluid and protonic superconductor. We propose that this system possesses vortex lines. The topological charge of the vortex lines are characterized by the Hopf indices and the Brower degrees of ϕ -mapping.

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I. INTRODUCTION

In the standard model for a neutron star its interior features superfluidity of neutron Cooper pairs and superconductivity of proton Cooper pairs (see,e.g.,[1, 2]). Both these condensates allow vortices of the $S^1 \rightarrow S^1$ map. It was suggested that the phenomenon of glitches in Crab and Vela pulsar is connected with vortex matter in these stars [3]. The neutron star spin down can be explained by neutrino radiation from superfluid vortex [4]. This remains a topic of intensive studies and discussions (for recent developments and citations see [5][6]). Besides that a standard model for a neutron star is a special system being a two component superfluid Bose condensed system which makes it also being a topic of abstract academic interest [7] since such a system allows for interesting phenomena with no direct counterparts in e.g. superconducting metals. Since the two component superfluid Bose condensed system have played an important roles in the condensed matter and neutron stars, studies of topological defects in this system becomes an important study aspect [8].

In this paper we mainly focus attention on the two component superfluid Bose condensed system, which is believed to be realized in the interior of neutron stars. We start by the free energy density of the two component superfluid Bose condensed system. By making use of Duan-Ge's decomposition theory of gauge potential and the Duan's topological current theory proposed by Prof. Duan Yi-Shi [9], we propose that this system possesses vortex lines, and the topological charge of the vortex lines are characterized by the Hopf indices and the Brower degrees of ϕ -mapping.

II. TWO COMPONENT BOSE CONDENSED SYSTEM

We consider the following effective Landau-Ginzburg free energy density that describes a two component superfluid Bose condensed system. In this system, we have a proton condensate described by ψ_1 and a neutron condensate described by ψ_2 . We do not consider the normal component of the protons and neutrons with their specific excitations, only the superfluid component. The ψ_1 field with electric charge q , which is actually twice the fundamental charge of the proton $q = 2|e|$, interacts with the gauge field \vec{A} , with $\vec{B} = \nabla \times \vec{A}$. The free energy density reads the following

expression[10]:

$$f = \frac{\hbar^2}{2M_1} \left| \left(\nabla - \frac{iq}{\hbar c} \vec{A} \right) \psi_1 \right|^2 + \frac{\hbar^2}{2M_2} |\nabla \psi_2|^2 + \frac{\vec{B}^2}{8\pi} + V(|\psi_1|^2, |\psi_2|^2), \quad (1)$$

where $M_1 = 2m_1$ and m_1 is the mass of the proton, and $M_2 = 2m_2$ and m_2 is the mass of the neutron. In the free energy density given above, we have ignored the term coupling the proton and neutron superfluid velocities, which gives rise to the Andreev-Bashkin effect[11], as it is not important in our discussion. Indeed, the relevant term in the free energy can be represented as $\sim \int d^3x \vec{v}_1 \cdot \vec{v}_2$, where \vec{v}_1 and \vec{v}_2 are velocities of the superfluid components. For neutron stars which do not rotate (the case which is considered in this paper), $\vec{v}_2 = 0$, and the effect obviously vanishes. We expect that due to the small density of the neutron vortices (compared to the density of the proton vortices) the effect is still negligible for most of the flux tubes in a rotating star as well [10]. The effect could be important only for a few of the flux tubes situated close to a neutron vortex core, where \vec{v}_2 strongly deviates from the constant value at interflux distance scales. We have also ignored the fact that the neutron condensate has a nontrivial 3p_2 order parameter as only the magnitude of the neutron condensate is relevant to the effect described below. The free energy density (1) only describes large distances and it does not describe the gap structure on the Fermi surfaces, only the superfluid component of the protons and neutrons.

For discussing the detail of Eq. (1), we start by introducing variables χ_1, χ_2 and ρ by

$$\psi_1 = \sqrt{2M_1\rho}\chi_1, \quad \psi_2 = \sqrt{2M_2\rho}\chi_2 \quad (2)$$

where the complex $\chi_1 = |\chi_1|e^{i\phi_1}$ and $\chi_2 = |\chi_2|e^{i\phi_2}$ are chosen so that $|\chi_1|^2 + |\chi_2|^2 = 1$. The modulus ρ then has the following expression:

$$\rho^2 = \frac{1}{2} \left[\frac{|\psi_1|^2}{M_1} + \frac{|\psi_2|^2}{M_2} \right]. \quad (3)$$

By putting formula (2) to use, the free energy density (1) can be represented as

$$f = \hbar^2 (\nabla \rho)^2 + \hbar^2 \rho^2 \left| \left(\nabla - \frac{iq}{\hbar c} \vec{A} \right) \chi_1 \right|^2 + \hbar^2 \rho^2 |\nabla \chi_2|^2 + \frac{\vec{B}^2}{8\pi} + V. \quad (4)$$

Then we can introduce current density

$$\vec{J} = \frac{i\hbar}{2M_1} (\psi_1 \nabla \psi_1^* - \psi_1^* \nabla \psi_1) + \frac{i\hbar}{2M_2} (\psi_2 \nabla \psi_2^* - \psi_2^* \nabla \psi_2) - \frac{q}{M_1 c} \vec{A} |\psi_1|^2 \quad (5)$$

and electrical current density

$$\vec{J}_q = \frac{i\hbar q}{2M_1}(\psi_1 \nabla \psi_1^* - \psi_1^* \nabla \psi_1) - \frac{q^2}{M_1 c} \vec{A} |\psi_1|^2 \quad (6)$$

Then we can get

$$\vec{J} = \hbar \rho^2 \vec{j} - \frac{2\rho^2 q}{c} \vec{A} |\chi_1|^2, \quad (7)$$

where

$$\vec{j} = i(\chi_1 \nabla \chi_1^* - \chi_1^* \nabla \chi_1 + \chi_2 \nabla \chi_2^* - \chi_2^* \nabla \chi_2). \quad (8)$$

We introduce a gauge-invariant vector field \vec{C} , directly related to the current density by

$$\vec{C} = \frac{\vec{J}}{\rho^2}. \quad (9)$$

We can also get the electrical current density

$$\vec{J}_q = \hbar q \rho^2 [i(\chi_1 \nabla \chi_1^* - \chi_1^* \nabla \chi_1) - \frac{2q}{c\hbar} \vec{A} |\chi_1|^2]. \quad (10)$$

We then rearrange the terms in Eq. (4) as follows: we add and subtract from Eq. (4) a term $\frac{1}{4} \hbar^2 \rho^2 \vec{j}$ and observe that the following expression:

$$\hbar^2 \rho^2 [|\nabla \chi_1|^2 + |\nabla \chi_2|^2 - \frac{\vec{j}^2}{4}] \quad (11)$$

is also gauge invariant. Indeed if we introduce the gauge invariant field

$$\vec{n} = \bar{\chi} \vec{\sigma} \chi, \quad (12)$$

where $\bar{\chi} = (\chi_1^*, \chi_2^*)$ and $\vec{\sigma}$ are Pauli matrices, then \vec{n} is a unit vector $|\vec{n}| = 1$ and we can write Eq. (11) as follows:

$$\hbar^2 \rho^2 [|\nabla \chi_1|^2 + |\nabla \chi_2|^2 - \frac{\vec{j}^2}{4}] = \frac{1}{4} \hbar^2 \rho^2 (\nabla \vec{n})^2. \quad (13)$$

Now we consider the remaining terms in Eq. (4). For the magnetic field we can get

$$\vec{B} = \nabla \times \vec{A} = \frac{\hbar c}{2q |\chi_1|^2} (\nabla \times \vec{j} - \frac{1}{\hbar} \nabla \times \vec{C}) \quad (14)$$

where $\nabla \times \vec{j}$ can be rewritten by the unit vector \vec{n} as following expression:

$$\nabla \times \vec{j} = \frac{1}{2} \vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n}. \quad (15)$$

Combining these we can obtain our main result: the free energy density becomes

$$\begin{aligned}
f = & \hbar^2(\nabla\rho)^2 + \frac{1}{4}\hbar^2\rho^2(\nabla\vec{n})^2 \\
& + \frac{\hbar^2c^2}{128\pi q^2|\chi_1|^4}\left[\frac{1}{\hbar}(\partial_\mu C_\nu - \partial_\nu C_\mu) - \vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n}\right]^2 \\
& - \frac{\hbar}{2q|\chi_1|^2}\vec{J}_q \cdot \left(\vec{J} - \frac{\vec{C}}{\hbar}\right) + \frac{1}{4}\hbar^2\rho^2\left[\vec{J}^2 - \frac{1}{|\chi_1|^2}\left(\vec{J} - \frac{\vec{C}}{\hbar}\right)^2\right] \\
& + V(\rho, |\chi_1|^2, |\chi_2|^2).
\end{aligned} \tag{16}$$

Let us discuss topological defects, allowed in Eq. (16).

III. TOPOLOGICAL VORTEX LINES IN THE TWO COMPONENT BOSE CONDENSED SYSTEM

As shown in Eq. (16), we know that the magnetic field of the system can be divided into two parts: One is the contribution of field C_μ , we learn that this part is introduced by the current density and can only present us with the topological defects as what in a single-condensate system. The other part, the contribution $\vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n}$ to the magnetic field term in Eq. (16), is a fundamentally topological property of this system. Here we emphasize that there allow another nontrivial topological configurations, which are originated from the contribution of the term $\vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n}$. Thus we will investigate this term in detail.

It is easy to prove that this term $\vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n}$ can be reexpressed in an Abelian field tensor form

$$\vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n} = \partial_\mu b_\nu - \partial_\nu b_\mu \tag{17}$$

where b_μ is the Wu-Yang potential [12]

$$b_\mu = \vec{e}_1 \cdot \partial_\mu \vec{e}_2 \tag{18}$$

Here \vec{e}_1 and \vec{e}_2 are two perpendicular unit vectors normal to \vec{n} , and $(\vec{e}_1, \vec{e}_2, \vec{n})$ forms an orthogonal frame:

$$\begin{aligned}
\vec{e}_1 \cdot \vec{e}_2 &= \vec{e}_1 \cdot \vec{n} = \vec{e}_2 \cdot \vec{n} = 0, \\
\vec{e}_1 \cdot \vec{e}_1 &= \vec{e}_2 \cdot \vec{e}_2 = \vec{n} \cdot \vec{n} = 1
\end{aligned} \tag{19}$$

Now, consider a two-component vector field $\vec{\phi} = (\phi^1, \phi^2)$ residing in the plane formed by \vec{e}_1 and \vec{e}_2 :

$$e_1^a = \frac{\phi^a}{\|\phi\|} \quad e_2^a = \epsilon^{ab} \frac{\phi^b}{\|\phi\|} \quad (\|\phi\|^2 = \phi^a \phi^a; a, b = 1, 2) \tag{20}$$

It can be proved that relations for \vec{e}_1 and \vec{e}_2 satisfy above restriction (19). Obviously the zero point of $\vec{\phi}$ are just the two-dimensional singular points of \vec{e}_1 and \vec{e}_2 .

Using the $\vec{\phi}$ field, the Wu-Yang potential can be expressed as

$$b_\mu = \epsilon_{ab} \frac{\phi^a}{\|\phi\|} \partial_\mu \frac{\phi^b}{\|\phi\|}. \quad (21)$$

Comparing with the expression of Duan-Ge's decomposition theory of gauge potential [13], we learn that b_μ satisfies the $U(1)$ gauge transformation. We introduce a two order tensor

$$H_{\mu\nu} = \vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n} \quad (22)$$

which describes the magnetic field that becomes induced in the system due to a non-trivial electromagnetic interaction in this system.

By using the $\vec{\phi}$ field, the two order tensor $H_{\mu\nu}$ can be reexpressed as

$$H_{\mu\nu} = 2\epsilon_{ab} \partial_\mu \frac{\phi^a}{\|\phi\|} \partial_\nu \frac{\phi^b}{\|\phi\|} \quad (23)$$

Because this tensor $H_{\mu\nu}$ plays an important role in the topological feature in this system, here we will use the Duan's topological current theory, to research the properties hidden in this tensor. To do so, we introduce a topological tensor current $K^{\mu\nu}$ [13], which is denoted by

$$K^{\mu\nu} = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda\rho} H_{\lambda\rho} = \frac{1}{4\pi} \epsilon^{\mu\nu\lambda\rho} \epsilon_{ab} \partial_\mu \frac{\phi^a}{\|\phi\|} \partial_\nu \frac{\phi^b}{\|\phi\|}. \quad (24)$$

According to [9] and using $\partial_\mu \frac{\phi^a}{\|\phi\|} = \frac{\partial_\mu \phi^a}{\|\phi\|} + \phi^a \partial_\mu \frac{1}{\|\phi\|}$ and the Green function relation in ϕ -space:

$$\partial_a \partial_a \ln \|\phi\| = 2\pi \delta^2(\vec{\phi}) \quad (\partial_a = \frac{\partial}{\partial \phi^a}), \quad (25)$$

it can be proved that

$$K^{\mu\nu} = \delta^2(\vec{\phi}) D^{\mu\nu}(\frac{\phi}{x}) \quad (26)$$

where $D^{\mu\nu}(\frac{\phi}{x}) = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{ab} \partial_\lambda \phi^a \partial_\rho \phi^b$ is the Jacobian between the ϕ space and the Cartesian coordinates. Denoting the spacial component of $K^{\mu\nu}$ by K^i , we obtain

$$K^i = K^{0i} = \delta^2(\vec{\phi}) D^i(\frac{\phi}{x}) \quad (i = 1, 2, 3) \quad (27)$$

where $D^i(\frac{\phi}{x}) = D^{0i}(\frac{\phi}{x})$.

An important conclusion from formula (27) is

$$\begin{cases} K^i = 0 & \text{if and only if } \vec{\phi} \neq 0 \\ K^i \neq 0 & \text{if and only if } \vec{\phi} = 0. \end{cases} \quad (28)$$

So it is necessary to study the zero point of ϕ to determine the nonzero solution of K^i . The implicit function theory [14] show that under the regular condition $D^i(\frac{\phi}{x}) \neq 0$, the general solution of

$$\phi^1(t, x^1, x^2, x^3) = 0, \quad \phi^2(t, x^1, x^2, x^3) = 0 \quad (29)$$

Can be expressed as

$$\begin{aligned} x^1 &= x_k^1(s, t) & x^2 &= x_k^2(s, t) \\ x^3 &= x_k^3(s, t) & (k &= 1, 2, 3, \dots, N) \end{aligned} \quad (30)$$

which represent the world surfaces of N moving isolated singular strings L_k with string parameter s . This indicates that in this system, there are vortex lines located at the zero points of the $\vec{\phi}$ field.

Now we will study the topological charges of the vortex lines and their properties by making using of the Duan's topological current theory. In δ -function theory[15], one can prove that in three-dimensional space

$$\delta^2(\vec{\phi}) = \sum_{k=1}^N \beta_k \int_{L_k} \frac{\delta^3(\vec{x} - \vec{x}_k(s))}{|D(\frac{\phi}{u})|_{\Sigma_k}} \quad (31)$$

where

$$D(\frac{\phi}{u})_{\Sigma_k} = \frac{1}{2} \epsilon^{\mu\nu} \epsilon_{mn} (\frac{\partial \phi^m}{\partial u^\mu}) (\frac{\partial \phi^n}{\partial u^\nu}) \quad (32)$$

and Σ_k is the k th planar element transverse to L_k with local coordinates (u^1, u^2) . The positive integer β_k is the Hopf index of ϕ -mapping, which means that when the point \vec{x} covers the neighborhood of the zero point \vec{x}_k once. The vector field $\vec{\phi}$ covers the corresponding region in ϕ -space β_k times. Meanwhile, the direction vector of L_k is given by [9]

$$\left. \frac{dx^i}{ds} \right|_{\vec{x}_k} = \left. \frac{D^i(\phi/x)}{D(\phi/u)} \right|_{\vec{x}_k}, \quad (33)$$

which leads to

$$\left. \frac{dx^1}{dx^3} \right|_{\vec{x}_k} = \left. \frac{D^1(\phi/x)}{D^3(\phi/x)} \right|_{\vec{x}_k}, \quad \left. \frac{dx^2}{dx^3} \right|_{\vec{x}_k} = \left. \frac{D^2(\phi/x)}{D^3(\phi/x)} \right|_{\vec{x}_k}. \quad (34)$$

There from (31) and (33). We find the inner topological structure of

$$K^i = \delta^2(\vec{\phi}) D^i(\phi/x) = \sum_{k=1}^N \beta_k \eta_k \int_{L_k} \frac{dx^i}{ds} \delta^3(\vec{x} - \vec{x}_k(s)) ds \quad (35)$$

in which η_k is the Brouwer degree of the ϕ -mapping, with

$$\eta_k = \text{sgn} D(\phi/u)_{\Sigma_k} = \pm 1. \quad (36)$$

Hence the topological charge of the vortex line L_k is

$$Q_k = \int_{\Sigma_k} K^i d\sigma_i = W_k, \quad (37)$$

where $W_k = \beta_k \eta_k$ is the winding number of $\vec{\phi}$ around L_k . And the total topological number on surface Σ is

$$Q = \int_{\Sigma} K^i d\sigma_i = \sum_{k=1}^n W_k. \quad (38)$$

Then we come the conclusion: under the regular condition $D^i(\phi/x) \neq 0$ there exist vortex lines in this system, whose topological charge are just the winding numbers of the ϕ -mapping.

IV. CONCLUSION

In conclusion, based on the Duan-Ge's decomposition theory of gauge potential and the Duan's topological current theory, we point out there allow vortex lines in the two component superfluid Bose condensed system, which is believed to be realized in the interior of neutron stars. Under the regular condition $D^i(\phi/x) \neq 0$, vortex lines are originated from the zero points of the $\vec{\phi}$ field and the topological charges of vortex lines are characterized by the winding numbers of the ϕ -mapping.

In this paper we treat the vortex lines as mathematical lines, i.e., the width of a line is zero. This description is obtained in the approximation that the radius of curvatures of a line is much larger than the width of the line [16].

At last, it should be pointed out that in the present paper, we just considered the case in which the neutron star does not rotate, but the effect of the neutron star's rotation is significant. We will study the topological aspects of rotary neutron star in our next work.

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